q–DEFORMED STAR PRODUCTS AND MOYAL BRACKETS

Ömer F. DAYI[1],

TÜBİTAK-Marmara Research Centre, Research Institute for Basic Sciences, Department of Physics, P.O.Box 21, 41470 Gebze, Turkey.

Abstract

The standard and anti–standard ordered operators acting on two–dimensional q–deformed phase space are shown to satisfy algebras which can be called q–W_∞. q–star products and q–Moyal brackets corresponding to these algebras are constructed. Some applications like defining q–classical mechanics and q–path integrals are discussed.

---

[1]E-mail address: dayi@mam.gov.tr.

1 Introduction

A way of visualizing quantum mechanics as $\hbar$–deformation of the classical case is to utilize symbols of operators, star products and Moyal brackets[1]. If we denote the symbol map by $\mathcal{S}_O^{(\hbar)}$, to an operator $\hat{f}$ there corresponds a c–number object $f_O^{(\hbar)}$:

$$\mathcal{S}_O^{(\hbar)}\left(\hat{f}\right) = f_O^{(\hbar)},$$

where the subscript O denotes the operator ordering adopted. Non-commutativity of quantum mechanics is taken into account in terms of the star product defined such that

$$\mathcal{S}_O^{(\hbar)}\left(\hat{f}\hat{g}\right) = \mathcal{S}_O^{(\hbar)}\left(\hat{f}\right) \star_O^{\hbar} \mathcal{S}_O^{(\hbar)}\left(\hat{g}\right)$$

and the symbol corresponding to the commutator (divided by $\hbar$) of the operators $\hat{f}$ and $\hat{g}$ is the Moyal bracket (or $\star^{\hbar}$–bracket):

$$\{f_O^{(\hbar)}(p,x), g_O^{(\hbar)}(p,x)\}_M \equiv \frac{1}{i\hbar}\left(f_O^{(\hbar)}(p,x) \star_O^{\hbar} g_O^{(\hbar)}(p,x) - g_O^{(\hbar)}(p,x) \star_O^{\hbar} f_O^{(\hbar)}(p,x)\right), \tag{1}$$

where p and x are classical phase space variables.

Recently $\star$–products and Moyal brackets are used in some diverse context like constructing a representation of non–commuting forms[2] and they appeared in the formulation of two–dimensional area preserving diffeomorphisms[3]. The latter is closely related to the quantum Hall effect which also exhibits a q–deformed structure where q is related to filling fraction[4]. Hence, understanding q–deformations in terms of symbols of the q–deformed operators may shed light on some aspects of the quantum Hall effect. Although the quantum Hall effect is a many body problem, it is based on a field theory which is given in a two–dimensional phase space when it is restricted to the lowest Landau level. Hence, understanding q–deformed star products and Moyal brackets in a two–dimensional phase space is the first step in this direction.

One of the essential properties of the ordinary star product is its associativity. In this case it is known that the unique deformation of the Poisson bracket is the Moyal bracket (1)[5]. On the other hand it was shown that it is impossible to deforme Heisenberg dynamics by keeping the algebra of the observables associative[6]. Hence, to obtain a nontrivial deformation of the quantum mechanics we should sacrifice associativity of the star product. In fact, q–deformed algebras are not associative, so that a q–deformed star product leading to them should also be non-associative.

Here, after deriving the algebras of the standard (or XP) and the anti–standard (or PX) ordered operators q–deformed star products and Moyal brackets corresponding to them are constructed and the obstacles in generalizing Weyl ordering to the q–deformed operators are emphasized.

Once a q–Moyal bracket is obtained an immediate application is to define q–classical mechanics in terms of q–Poisson brackets. This and some other applications like classical as well as quantum mechanical properties of q–canonical transformations and a definition of q–path integrals on general grounds are briefly discussed.

2 Ordinary Symbols and Star Products:

In the two–dimensional phase space given in terms of the ordinary ($\hbar$–deformed) canonical operators

$$[P_\hbar, X_\hbar] = i\hbar, \tag{2}$$

an operator $\mathcal{O}$ can be written as

$$\mathcal{O} = \sum_{(m,n)>0} \mathcal{O}_{m,n} g(P_\hbar^m, X_\hbar^n), \tag{3}$$

where $g(P_\hbar^m, X_\hbar^n)$ are some functions of $P_\hbar^m, X_\hbar^n$ and $\mathcal{O}_{m,n}$ are some constant coefficients depending on the operator ordering scheme adopted. Hence, it is sufficient to deal with the monomials $g(P_\hbar^m, X_\hbar^n)$ as far as the algebraic properties of the operators are concerned. We are interested in the following three different ordering schemes which constitute complete basis.

Standard (or XP) ordering: In this scheme one deals with the monomials $X_\hbar^n P_\hbar^m$. They satisfy the Lie algebra[7]

$$[X_\hbar^n P_\hbar^m, X_\hbar^k P_\hbar^l] = \sum_{r=1}^{\infty} (i\hbar)^r r! \left\{ \binom{k}{r}\binom{m}{r} - \binom{n}{r}\binom{l}{r} \right\} X_\hbar^{n+k-r} P_\hbar^{l+m-r}. \tag{4}$$

Antistandard (or PX) ordering: the suitable monomials $P_\hbar^n X_\hbar^m$, can be shown to satisfy[7]

$$[P_\hbar^n X_\hbar^m, P_\hbar^k X_\hbar^l] = \sum_{r=1}^{\infty} (-i\hbar)^r r! \left\{ \binom{k}{r}\binom{m}{r} - \binom{n}{r}\binom{l}{r} \right\} P_\hbar^{n+k-r} X_\hbar^{l+m-r}. \tag{5}$$

The algebras (4) and (5) are called W_∞[3].

Weyl ordering: The monomials

$$T_{m,n}^{(\hbar)} = \exp(P_\hbar \frac{\partial}{\partial \mathcal{P}} + X_\hbar \frac{\partial}{\partial \mathcal{Q}}) \mathcal{P}^m \mathcal{Q}^n |_{\mathcal{P}=\mathcal{Q}=0}, \tag{6}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are c–number variables, constitute a complete basis and satisfy the Lie algebra[8],[9]

$$[T_{m,n}^{(\hbar)}, T_{k,l}^{(\hbar)}] = \sum_{a=0}^{b} (i\hbar)^{2a+1} B_{mnkl}^a T_{m+k-2a-1,n+l-2a-1}^{(\hbar)}, \tag{7}$$

where $b = \min\{[(m+k-1)/2, (n+l-1)/2]\}$. $B_{00kl} = B_{mn00} \equiv 0$ and for the other values of the indices

$$B_{mnkl}^a \equiv \sum_{c=0}^{2a+1} \frac{(-1)^c m! n! k! l!}{(2a+1-b)! b! (m+c-2a-1)!(n-c)!(k-c)!(l+c-2a-1)!}. \tag{8}$$

For our purposes it is sufficient to deal with the symbols of the monomials in each ordering scheme, which are given by

$$\mathcal{S}_S^{(\hbar)}[X_\hbar^m P_\hbar^n] = \mathcal{S}_A^{(\hbar)}[P_\hbar^m X_\hbar^n] = \mathcal{S}_W^{(\hbar)}[T_{m,n}^{(\hbar)}] = p^m x^n, \tag{9}$$

where p and x are c–number variables.

Related star products are[1],[7]

$$\star_S^\hbar \equiv \exp\left(i\hbar \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right), \tag{10}$$

$$\star_A^\hbar \equiv \exp\left(-i\hbar \overleftarrow{\partial}_x \overrightarrow{\partial}_p\right), \tag{11}$$

$$\star_W^\hbar \equiv \exp\left[\frac{-i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\right]. \tag{12}$$

We used $\frac{\partial}{\partial x} \equiv \partial_x$; $\frac{\partial}{\partial p} \equiv \partial_p$. One of them can be utilized in the Moyal bracket (1) in terms of the related symbols. Observe that $\star^\hbar$–products are associative[1], so that the algebraic properties of the commutators are preserved.

3 Classical ($\hbar = 0$) q–Deformed Symbols, Star Products and Moyal Brackets:

Classical ($\hbar = 0$) q–deformed canonical operators are defined as

$$P_q X_q - q X_q P_q = 0. \tag{13}$$

4

Now, the monomials of the different ordering schemes are equivalent up to an overall q–dependent factor. Thus it is sufficient to consider the algebra

$$q^{nk}T_{m,n}^{(q)}T_{k,l}^{(q)} - q^{ml}T_{k,l}^{(q)}T_{m,n}^{(q)} = 0, \tag{14}$$

where $T_{m,n}^{(q)} \equiv P_q^m X_q^n$. They give a complete basis in P_q, X_q space and the basis operators in another ordering scheme are equal to $T_{m,n}^{(q)}$ up to an overall q–dependent constant.

Symbols of the operators $T_{m,n}^{(q)}$ are

$$\mathcal{S}^{(q)}T_{m,n}^{(q)} = p^m x^n. \tag{15}$$

It is possible to define associative $\star^q$–products

$$\star_S^q \equiv \exp\left(\nu \overleftarrow{\partial}_p \, px \, \overrightarrow{\partial}_x\right), \tag{16}$$

$$\star_A^q \equiv \exp\left(-\nu \overleftarrow{\partial}_x \, xp \, \overrightarrow{\partial}_p\right), \tag{17}$$

$$\star_W^q \equiv \exp\left[\frac{-\nu}{2}(\overleftarrow{\partial}_x \, xp \, \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \, px \, \overrightarrow{\partial}_x)\right], \tag{18}$$

where $\nu \equiv \ln q$ and the subscripts denote the resemblance to the ordinary star products (10)–(12). In fact, (16)–(18) can be obtained from (10)–(12) by a canonical transformation in accord with the fact that there exists a unique deformation of the Poisson bracket if the star product is assocaitive.

The symbols defined in (15) satisfy

$$q^{nk}p^m x^n \star^q p^k x^l - q^{ml}p^k x^l \star^q p^m x^n = 0, \tag{19}$$

for any $\star^q$–product (16)–(18).

4 q–symbols, q–star Products and q–Moyal Brackets:

q–deformed ($\hbar \neq 0$) canonical variables satisfy

$$PX - qXP = i\hbar. \tag{20}$$

To reproduce (14) in $\hbar = 0$ limit we define the q–commutator as

$$[t(P^m, X^n), t(P^k, X^l)]_q \equiv q^{nk}t(P^m, X^n)t(P^k, X^l) - q^{ml}t(P^k, X^l)t(P^m, X^n), \tag{21}$$

where $t(P^m, X^n)$ is a function of P^m and X^n, depending on the operator ordering scheme adopted. Observe that the weights of the q–commutator (21) change according to the operators which are considered.

As one can easily see, the standard or the anti–standard ordered monomials form a complete basis in the q–phase space given by P and X.

By explicit calculations q–algebra satisfied by the standard ordered monomials $X^n P^m$ can be derived:

$$[X^n P^m, X^k P^l]_q = \sum_{r=1}^{\infty} (i\hbar)^r [r]! \left\{ q^{(k-r)(n-r)+ml} \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} m \\ r \end{bmatrix} - q^{(m-r)(l-r)+nk} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} l \\ r \end{bmatrix} \right\}$$
$$X^{n+k-r} P^{l+m-r}, \tag{22}$$

where we used the definitions of the q–factorial

$$[n]! \equiv \left(\frac{1-q^n}{1-q} \right)! \equiv [1][2] \cdots [n-1][n],$$

and the q–binomial coefficient

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \frac{[n]!}{[n-r]![r]!}.$$

Similarly we see that in the anti–standard ordering scheme the monomials $P^m X^n$ satisfy the q–algebra

$$[P^n X^m, P^k X^l]_q = \sum_{r=1}^{\infty} (-i\hbar)^r q^{r(r-1)/2} [r]! \left\{ \begin{bmatrix} k \\ r \end{bmatrix} \begin{bmatrix} m \\ r \end{bmatrix} - \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} l \\ r \end{bmatrix} \right\} P^{n+k-r} X^{m+l-r}.$$
$$\tag{23}$$

The q–deformed algebras (22) and (23) can be called q–W_∞. For other definitions of q–W_∞ algebra see [10].

Symbol maps for the standard and anti-standard orderings are defined like the ordinary case:

$$\mathcal{S}_S(X^m P^n) = \mathcal{S}_A(P^m X^n) = p^m x^n. \tag{24}$$

As emphasized before, the ordinary star products should be associative, so that, the Moyal bracket satisfies an identity corresponding to the Jacobi identity satisfied by the ordinary operators. However, now the situation is altered drastically: we do not any more deal with the commutators which are not aware of their entries but with the q–commutators (21) which change according to their entries i.e. the underlying algebraic structure is non-associative. Thus the associativity condition cannot be preserved and we can obtain a non-trivial deformation of the Poisson bracket other than $\hbar$–deformation.

In terms of the q–derivative

$$D_z f(z) \equiv \frac{f(z) - f(qz)}{(1-q)z}, \tag{25}$$

we can construct q–star products for the standard and anti–standard orderings as

$$\star_S \equiv \sum_{r=0}^{\infty} \frac{(i\hbar)^r}{[r]!} \overleftarrow{D}_p^r \exp(\nu \overleftarrow{\partial}_p \, px \, \overrightarrow{\partial}_x) \overrightarrow{D}_x^r, \tag{26}$$

$$\star_A \equiv \sum_{s=0}^{\infty} (-\nu \overleftarrow{\partial}_x \, x)^s \sum_{r=0}^{\infty} \frac{(-i\hbar)^r q^{r(r-1)/2}}{[r]!} \overleftarrow{D}_x^r \overrightarrow{D}_p^r (p \overrightarrow{\partial}_p)^s. \tag{27}$$

One can see that if the q–Moyal bracket is defined as

$$\{p^m x^n, p^k x^l\}_{q-M} \equiv \frac{1}{i\hbar}(q^{nk} p^m x^n \star p^k x^l - q^{ml} p^k x^l \star p^m x^n), \tag{28}$$

by using the symbols (24) and the q–star products (26)–(27), q–Moyal algebras corresponding to (22) and (23) are obtained. In [11] another q–star product is defined by using the coherent states maps, however it does not lead to the algebras which we deal with (22),(23).

Generalization of the Weyl ordering (6) to the q–phase space is not obvious: a term of a monomial can be generalized by assuming that it is weighted with a factor q^γ, where γ is a number depending on the term under consideration. To emphasize the difficulties related to this ordering procedure, let us suppose that there exist operators $T_{m,n}$ leading to the ordinary Weyl ordered operators in the $q = 1$ limit, satisfying

$$[T_{m,n}, T_{k,l}]_q = \sum_{r,s=0}^{A,B} \mathcal{C}^{rs}_{mnkl}(\hbar, q) T_{r,s}, \tag{29}$$

where for obtaining the correct classical limit $\mathcal{C}$ should satisfy

$$\mathcal{C}^{(K)rs}_{mnkl}(0, q) = 0. \tag{30}$$

An operator algebra is proposed in [9] as a generalization of (7) by replacing the factorial terms with the q–factorials:

$$[T^{(GF)}_{m,n}, T^{(GF)}_{k,l}]_q = \sum_{a=0}^{b} \sum_{c=0}^{2a+1} \frac{(-1)^c [m]![n]![k]![l]!}{[2a+1-b]![b]![m+c-2a-1]![n-c]![k-c]![l+c-2a-1]!}$$
$$(i\hbar)^{2a+1} T^{(GF)}_{m+k-2a-1,n+l-2a-1}, \tag{31}$$

7

where b is given as in (7). Symbol map independent of the definition of the q–Weyl ordered operators is

$$\mathcal{S}_W T_{m,n}^{(GF)} = p^m x^n, \tag{32}$$

so that, the $\star$–product reproducing (31), in terms of the q–Moyal bracket (28), is

$$\star_W \equiv \sum_{M=0}^{\infty} (-\nu/2)^M \sum_{L=0}^{M} \frac{(-1)^L}{(M-L)!L!} (\overleftarrow{\partial}_x x)^{M-L} (\overleftarrow{\partial}_p p)^L \sum_{\alpha=0}^{\infty} (-i\hbar/2)^{\alpha}$$
$$\sum_{\beta=0}^{\alpha} \frac{(-1)^{\beta}}{[\alpha-\beta]![\beta]!} \overleftarrow{D}_x^{\alpha-\beta} \overleftarrow{D}_p^{\beta} \overrightarrow{D}_p^{\alpha-\beta} \overrightarrow{D}_x^{\beta} (p \overrightarrow{\partial}_p)^{M-L} (x \overrightarrow{\partial}_x)^L. \tag{33}$$

Obviously, independent of how the generalization is done, we have

$$T_{m,0} = P^m, \ T_{0,m} = X^m. \tag{34}$$

Hence, by studying the q–commutators $[P^m, X^m]_q$ one can try to reach to the other q–weighted monomials: e.g.

$$P^2 X^2 - q^4 X^2 P^2 = i\hbar[2](PX + q^2 XP), \tag{35}$$

suggests that $T_{1,1} \approx (PX + q^2 XP)$. Similarly, explicit calculation gives

$$P^2 X^3 - q^6 X^3 P^2 = [2](PX^2 + q^4 X^2 P + q^2 XPX), \tag{36}$$

yielding $T_{1,2} \approx (PX^2 + q^4 X^2 P + q^2 XPX)$. However, q–commutator of this operator with P :

$$P(PX^2 + q^4 X^2 P + q^2 XPX) - q^2(PX^2 + q^4 X^2 P + q^2 XPX)P =$$
$$i\hbar(1 + q + q^2)(PX + q^3 XP), \tag{37}$$

suggests $T_{1,1} \approx (PX + q^3 XP)$, which is in contradiction with the one suggested by (35).

In fact, having obstacles in defining q–Weyl ordering is not surprising. If one considers monomials reproducing the Weyl ordered ones in the $q = 1$ limit with a definite q weight they can not constitute a complete basis. For having a complete basis one should have monomials with all possible q weights.

5 Applications:

In terms of the q–star products and the related q–Moyal brackets (26)–(28), one can study quantum as well as classical dynamics on general grounds.

If we deal with the $\star_S$–product, q–classical dynamics can be given in terms of the q–Poisson bracket defined for the observables $f(p,x) = \sum_i f_i(p,x)$ and $g(p,x) = \sum_j g_j(p,x)$ where f_i and g_j are monomials in p, x, as

$$
\begin{aligned}
\{f(x,p), g(x,p)\}_{q-P} &\equiv \lim_{\hbar \to 0} \{f(x,p), g(x,p)\}_{q-M} \\
&= \sum_{i,j} [q^{\alpha(f_i,g_j)} (D_p f_i) \exp(\nu \overleftarrow{\partial}_p \, px \, \overrightarrow{\partial}_x)(D_x g_j) \\
&\quad - q^{\alpha(g_j,f_i)} (D_p g_j) \exp(\nu \overleftarrow{\partial}_p \, px \, \overrightarrow{\partial}_x)(D_x f_i)],
\end{aligned}
\tag{38}
$$

where $\alpha(p^m x^n, p^k x^l) = nk$. Thus, if $H = \sum_k H_k$ is the classical hamiltonian where H_k are monomials, equation of motion of the observable $f(p,x)$ is

$$
\tau_q(f) = \{H, f\}_{q-P},
\tag{39}
$$

where $\lim_{q \to 1} \tau_q = d/dt$. Now, observe that in general

$$
\{H, fg\}_{q-P} \neq f\{H, g\}_{q-P} + g\{H, f\}_{q-P},
\tag{40}
$$

which leads to

$$
\tau_q(fg) \neq \tau(f)g + f\tau(g).
\tag{41}
$$

Obviously, there are some exceptions like f or g is constant. One may think that $\star_S$ of f and g should be considered on the left hand side of (41), however in the limit $\hbar \to 0$ the q–star product will yield $\star_S^{(q)}$ (16), and $f \star_S^{(q)} g = q^{\alpha(f,g)} fg$, so that there is not any difference.

In the ordinary classical mechanics canonical transformations leave the basic Poisson brackets invariant. One of these is the point transformation defined as

$$
u = f(x); \; p_u = (\partial_x f(x))^{-1} p,
\tag{42}
$$

where f is an invertible function. For the q–classical mechanics point transformation can be defined as

$$
u = f(x); \; p_u = (D_x f(x))^{-1} p.
\tag{43}
$$

Now, in terms of the q–Poisson bracket (38) one can observe that

$$
\{u, p_u\}_{q-P} = -q^{\alpha(u,p_u)}; \; \{p_u, u\}_{q-P} = 1,
\tag{44}
$$

instead of the ones satisfied by p, x :

$$
\{x, p\}_{q-P} = -q; \; \{p, x\}_{q-P} = 1.
\tag{45}
$$

In (44) $\alpha(u, p_u) = x\partial_x \log f(x)$ which is a number. Let us have an example where $f(x) = \sqrt{x}$:

$$u = \sqrt{x}; \ p_u = \left[\frac{1}{2}\right]^{-1} \sqrt{x} p, \qquad (46)$$

so that,

$$\{u, p_u\}_{q-P} = -q^{1/2}; \ \{p_u, u\}_{q-P} = 1. \qquad (47)$$

In fact, the transformation (46) was studied in [12] and found that it is a q–canonical transformation if the phase space operators satisfy

$$\hat{p}\hat{x} - q\hat{x}\hat{p} = i\hbar, \qquad (48)$$
$$\hat{p}_u\hat{u} - \sqrt{q}\hat{u}\hat{p}_u = i\hbar, \qquad (49)$$

which are consistent with (45) and (47).

When we deal with q–quantum mechanics in the Heisenberg picture, time evolution of an observable f is given by

$$\tau(f) = \{H, f\}_{q-M} = \frac{1}{i\hbar} \sum_{i,j} \left(q^{\alpha(H_i, f_j)} H_i \star f_j - q^{\alpha(f_j, H_i)} f_j \star H_i \right). \qquad (50)$$

Here $\star$ indicates one of the q–star products (26)–(27).

In the Schrödinger picture time evolution of a time–dependent state vector ψ is given by

$$i\hbar\tau\left(\psi(t)\right) = \hat{H}\psi(t), \qquad (51)$$

where $\hat{H}$ is the q–hamiltonian operator i.e. $\mathcal{S}(\hat{H}) = H$.

In contrary to the ordinary quantum mechanics, in the q–deformed case relation between the Schrödinger picture (51) and the Heisenberg picture (50) is not clear. The unique common feature of the deformed and non-deformed cases is the fact that in both of the cases symbols of the monomials are the same, namely $p^m x^n$. The difference lies in the definition of the related star products. Thus, we may still assume that the symbol of the evolution operator $\hat{U}(t)$ is

$$U(t) = \mathcal{S}\left(\hat{U}(t)\right) = e^{\frac{it}{\hbar}H}. \qquad (52)$$

Then we can adopt the definition of the path integral of the ordinary time evolution given in terms of the star products and symbols in [13] to define the q–path integral as

$$G(t) = \lim_{N \to \infty} U(\frac{t}{N}) \star \cdots \star U(\frac{t}{N}). \qquad (53)$$

When the canonical transformation (42) is performed there will be some q–quantum corrections in the transformed hamiltonian. The kinetic term including

the q–quantum corrections can be studied in terms of the $\star_S$–product similar to the ordinary case[7] by defining the transformed kinetic term as

$$\tilde{H}_0 = (D_x f)^{-1/2} \star_S p \star_S (D_x f)^{-2} \star_S D_x f \star_S p \star_S (D_x f)^{-1/2}. \qquad (54)$$

Because of the non-associativity of the q–star product we should specify in which order the multiplications will be performed in (54). However, we do not possess a general procedure.

As it is briefly illustrated, q–star products and q–Moyal brackets are very useful in studying several aspects of q–deformations on general grounds. However, the relations to the other formulations of q–dynamics (e.g. see [14]) and q–path integral definitions[15] should be studied.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. **111**, 61 (1978); **111**, 111 (1978).

[2] M. Reuter, Int. J. Mod. Phys A **11**, 1253 (1996).

[3] D.B. Fairlie and C.K. Zachos, Phys. Lett. B **224**, 101 (1989);
I. Bakas, Phys. Lett. B **228**, 57 (1989);
C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B **236**, 173 (1990);
J. Hoppe and P. Schaller, Phys. Lett. B **237**, 407 (1990).

[4] S. Iso, D. Karabali and B. Sakita, Phys. Lett. B **296**, 143 (1992);
A. Capelli, C.A. Trugenberger and G.R. Zemba, Nucl. Phys. B **396**, 465 (1993);
A. Capelli, C.A. Trugenberger and G.R. Zemba, Nucl. Phys. B **398**, 531

(1993);

I.I. Kogan, Int. J. Mod. Phys A **9**, 3887 (1994);

M. Flohr and R. Varnhagen, J. Phys. A: Math. Gen. **27**, 3999 (1994).

[5] W. Arveson, Commun. Math. Phys. **89**, 77 (1983);
P. Fletcher, Phys. Lett. B **248**, 323 (1990).

[6] M. Pillin, Commun. Math. Phys. **180**, 23 (1996).

[7] G.V. Dunne, J. Phys. A: Math. Gen. **21**, 2321 (1988).

[8] C.M. Bender and G.V. Dunne, Phys. Rev. D **40**, 3504 (1989).

[9] I.M. Gelfand and D.B. Fairlie, Commun. Math. Phys. **136**, 487 (1991).

[10] C-Z. Zha, J. Math. Phys. **35**, 517 (1994);
E.H. Al Kinani and M. Zakkari, Phys. Lett. B **357**, 105 (1995).

[11] A. Odzijewicz, *Quantum algebras and q–special functions related to coherent states maps of the disc*, Warsaw University preprint IFT–18/95.

[12] Ö.F. Dayi and I.H. Duru, J. Phys. A: Math. Gen. **28**, 2395 (1995).

[13] F.A. Berezin, Usp. Fiz. Nauk **132**, 497 (1980).

[14] A. Chodos and D.G. Caldi, J. Phys. A: Math. Gen. **24**, 5505 (1991).

[15] L. Baulieu and E.G. Floratos, Phys. Lett. B **258**, 171 (1991);
M. Chaichian and A.P. Demichev, Phys. Lett. B **320**, 273 (1994).